

# Incentivizing High-quality Content from Heterogeneous Users: On the Existence of Nash Equilibrium

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## Abstract

We study the existence of pure Nash equilibrium (PNE) for the mechanisms used in Internet services (e.g., online reviews and question-answer websites) to incentivize users to generate high-quality content. Most existing work assumes that users are homogeneous and have the same ability. However, real-world users are heterogeneous and their abilities can be very different from each other due to their diverse background, culture, and profession. In this work, we consider heterogeneous users with the following framework: (1) the users are heterogeneous and each of them has a private type indicating the best quality of the content she can generate; (2) there is a fixed amount of reward to allocate to the participated users. Under this framework, we study the existence of pure Nash equilibrium of several mechanisms composed by different allocation rules, action spaces, and information settings. We prove the existence of PNE for some mechanisms and the non-existence of PNE for some mechanisms. We also discuss how to find a PNE for those mechanisms with PNE either through a constructive way or a search algorithm.

## Introduction

More and more Internet websites rely on users' contribution to collect high-quality content, including knowledge-sharing services (e.g., Yahoo! Answers and Quora), online product commenting and rating services (e.g., Yelp, mobile app stores), and ecommerce websites (e.g., Amazon.com). For simplicity, we call websites that rely on User-Generated Content UGC websites. To attract more users and to incentivize them to contribute high-quality content, those sites usually give high-quality contributors some reward in the form of virtual value, which represents privilege and benefit, or monetary return, such as the gift card. To collect more reward, users usually strategically interact with those websites. Therefore, to maximize the quality of the content generated from users, a UGC website needs to carefully design their mechanisms and analyze users' behaviors. We call the mechanisms used by those UGC sites UGC mechanisms.

Recently, much effort has been devoted to the design and analysis of UGC mechanisms (Ghosh and Hummel 2011; Ghosh and McAfee 2011; Easley and Ghosh 2013). Most of those works assume that users are homogeneous - people

are of the same ability while contributing to the sites. However, in the real world, users' abilities can be very different from each other due to their diverse background, culture, and profession. For example, an experienced photographer can write a high-quality comment to a photo, which is very difficult for a non-experienced user. Thus, in this work, we study the game theoretical problem raised in Internet services with heterogeneous users. We introduce the concept of "type" for the problem, which denotes the ability of a user: the larger the type of a user is, the better content she can contribute to the site. We further assume that each user needs to afford a cost to participate in the game and contribute content. The cost reflects the effort of content generation (e.g. writing a review), such as time and mobile traffic. In our work, we assume users costs are bounded which is different from (Easley and Ghosh 2013; Ghosh and McAfee 2011). We believe that our setting is more practical because nobody would spend too much (if not infinite) effort to make contributions.

Two allocation rules are studied in our work: one is the top  $K$  allocation rule (Jain, Chen, and Parkes 2009; Easley and Ghosh 2013) in which users with the highest  $K$  qualities will get the reward equally. The other rule is proportional-share rule (Ghosh and McAfee 2011; Jain, Chen, and Parkes 2009; Nisan et al. 2007; Chen 2009), in which all the participants (who make non-zero contribution) will share the reward proportionally to their contributed qualities. The proportional share rule is also widely used in network rate control (Kelly 1997; Kelly, Maulloo, and Tan 1998), market allocation (Cachon and Lariviere 1999) and scheduling (Sandholm and Lai 2010; Stoica et al. 1996). Two action spaces are investigated: the binary action space, in which each user can only choose to participate in or not; and the continuous action space, in which each user can choose the quality of the content to contribute. Besides, we study the problem from both the full-information setting and the partial-information setting.

We study the existence of pure Nash equilibrium for several different UGC mechanisms by combining the above options. Our main results can be summarized as follows.

1. For the *full-information setting*, we prove the existence of PNE for the mechanism with the proportional allocation rule and the continuous action space. The key of the proof

is to construct a perturbed game, prove the existence of PNE for the perturbed game, and prove that the PNE of the perturbed game will converge to the equilibrium of the original game. We then discover several properties of the PNE of the mechanism, which are further used to design an algorithm to find a PNE for the mechanism. We also study three other mechanisms and show (1) the existence of PNE for the mechanism with the top  $K$  allocation rule and the binary action space and for the mechanism with the proportional allocation rule and the binary action space and (2) the non-existence of PNE for the mechanism with the top  $K$  allocation rule and the continuous action space.

2. For the *partial-information setting*, we prove the existence of a symmetric PNE for the mechanism with the top  $K$  allocation rule and the continuous action space. The key of the proof is to construct a simple but (maybe) infeasible symmetric strategy and then convert it to a feasible symmetric equilibrium strategy by repeated calibration. Our proof also provides a method to construct a symmetric PNE. For the binary action space, we prove the existence of equilibrium for the mechanisms with both top  $K$  allocation rule and proportional allocation rule.

## Related work

Recently UGC mechanisms have attracted much research attention (Anderson et al. 2013; Chawla, Hartline, and Sivan 2012; Ghosh and McAfee 2011; Ghosh and Hummel 2011; Ghosh and Kleinberg 2013). Most of the existing work concentrates on homogeneous settings, i.e. users are of the same ability to generate content for the website. (Ghosh and McAfee 2011) designs a simple voting rule under sequential and simultaneous model, in which both the quality of contributions and the number of contributors are endogenously determined. (Ghosh and Hummel 2011) studies the rank based allocation mechanism for websites with user-generated content and shows the mechanism always incentivizes higher quality equilibriums than the proportional allocation rule. (Ghosh and Kleinberg 2013) models the online education forums with two parameters which represent the frequency of checking forums by teachers and students separately. A brief survey about UGC mechanisms can be found in (Ghosh 2012).

The key differences between those existing literature and our work are that (1) we focus on heterogeneous users who have different abilities to generate content, and (2) we assume users' effort is bounded, i.e., users cannot afford an infinity cost to make contributions. One closely related work is (Easley and Ghosh 2013), which studies the problem of badge design. Although users' abilities are considered in that work, the participants are modeled as a continuum and as a result, the single user's behavior will not affect others' pay-off much. In this work, we regard users as discrete individuals, and one's strategy will impact others' utilities.

## Model

In this section, we describe the model for analyzing the incentives created by various UGC mechanisms, when contributors are strategic agents with heterogeneous abilities, and when the decision of whether to participate in and how much to contribute is a strategic choice.

There is a set of  $N$  strategic users in a UGC site, and each user  $i$  has a private type  $q_i \in [0, 1]$ , which indicates the best quality of the content the user can contribute to the site. Without loss of generality, we number the users according to the descending order of their types, i.e.  $q_1 \geq q_2 \geq \dots \geq q_N$ . Let  $x_i$  denote user  $i$ 's action, which indicates the quality of the content she actually contributes to the site. Note that we have  $0 \leq x_i \leq q_i$ . The user needs to afford a cost  $c_i$  for the action  $x_i$ . In this work, we consider linear cost for simplicity:  $c_i = c \frac{x_i}{q_i}$ , where  $c$  is a same upper bound<sup>1</sup> of the cost that a user can afford.

We study two action spaces in this work. The first one is a binary action space: each user can only choose not to contribute or to contribute content with quality  $q_i$  (i.e.  $x_i \in \{0, q_i\}$ ). The second one is a continuous action space: the quality  $x_i$  that user  $i$  contributes to the site is a continuous value between 0 and  $q_i$  (i.e.,  $x_i \in [0, q_i]$ ). Note sometimes we say that a user does not participate in the game if  $x_i = 0$ , and a user participates in the game if  $x_i > 0$ .

The site has a fixed number of reward  $R$  to allocate to the contributors, depending on their contributions. We study two allocation rules: the top  $K$  allocation (Jain, Chen, and Parkes 2009) and the proportional allocation (Ghosh and McAfee 2011). The first one allocates  $\frac{R}{K}$  to each user of those who contribute the top  $K$  largest qualities. Note that if  $N < K$ , each user can still only get  $\frac{R}{K}$  reward. The second one allocates the reward to all users proportional to their contributions: the reward  $r_i$  allocated to user  $i$  is  $\frac{x_i}{\sum_j x_j} R$  if  $x_i > 0$  and 0 if  $x_i = 0$ .

While analyzing the model, we consider two settings: the full-information setting and the partial-information setting. In the full-information setting, the types  $\{q_i\}_{i \in [N]}$  are deterministic and are known to all the users. In the partial-information setting, the type of each user is assumed to be drawn from a publicly known distribution  $F$ , the first order derivative of which is continuous, and each user only knows her own type  $q_i$ .

With the above notations, the utility of user  $i$  can be written as  $u_i(x_i, x_{-i}) = r_i(x_i, x_{-i}) - c_i(x_i)$ , where  $r(x_i, x_{-i})$  is the reward of user  $i$  given her strategy  $x_i$  and the strategies  $x_{-i}$  of other players. We assume that all the users are rational and they want to maximize their (expected) utilities.

## Full-information Setting

In this section, we study the mechanisms under the full-information setting. Recall that in this setting, the type  $q_i$  of any user is known to all the users. This setting corresponds to the real-world scenarios where the users are familiar with

<sup>1</sup>If each user has a different cost upper bound  $C_i$ , it is easy to absorb  $C_i$  into the private type  $q_i$  by scaling:  $q_i = \frac{q_i C_i}{C_i}$ .

each other. For example, considering a professional mathematical question posted in Yahoo! Answer, there will be only a few users in the Yahoo! Answer community who can answer the question and they know each other quite well.

Combining the different choices of the allocation rule and the action space, there are four mechanisms under this setting. We focus on the mechanism with the proportional allocation rule and the continuous action space here and directly list the results of the other three, which are relatively easy to analyze.

### $\mathcal{M}_1$ : Top $K$ Allocation, Binary Action Space

It is not difficult to see that PNE exists and is unique (except the case  $R = Kc$ ) for this scheme. There are three cases depending on the parameters  $(R, K, c)$ :

1. If  $R < Kc$ , no user will contribute content to the site and the equilibrium is  $x_i = 0, \forall i$ .
2. If  $R = Kc$ , there are many equilibria: any group of  $k \leq K$  users contributing to the site is an equilibrium.
3. If  $R > Kc$ , the first  $K$  users will contribute to the site. That is,  $x_i = q_i, \forall 1 \leq i \leq K$  and  $x_i = 0, \forall i > K$ .

### $\mathcal{M}_2$ : Top $K$ Allocation, Continuous Action Space

There does not exist PNE for this mechanism under the full-information setting according to the following discussions.

1. Zero-participating and more than  $K$  people participating in are obviously not an equilibrium.
2. If the equilibrium is constructed by fewer than  $K$  people participating in, there is at least one person would could get positive utility by making a positive contribution. This contradicts with the concept of equilibrium.
3. If there are  $K$  people participating in, all the contributors will generate contents with quality  $\epsilon \rightarrow 0$  but  $\epsilon \neq 0$ , so the equilibrium strategy does not exist.

### $\mathcal{M}_3$ : Proportional Allocation, Binary Action Space

It turns out that PNE exists and there can be multiple equilibria for this mechanism.

1. If  $R < c$ , there exists a unique equilibrium in which nobody will contribute:  $x_i = 0, \forall i$ .
2. If  $R = c$ , multiple equilibria exist: (1) nobody contributing is an equilibrium, and (2) any single user contributing is also an equilibrium.
3. If  $R > c$ , Nash equilibrium exists. Denote  $j$  as the index satisfying the following two inequalities:

$$\frac{Rq_j}{\sum_{k=1}^j q_k} > c,$$

and

$$\frac{Rq_{j+1}}{\sum_{k=1}^{j+1} q_k} \leq c.$$

Then  $x_i = q_i, \forall 1 \leq i \leq j$  and  $x_i = 0, \forall i > j$  compose an equilibrium. There can exist multiple equilibria. Consider an example with parameters  $N = 3, R = 4, c = 1, \mathbf{q} =$

$\{0.9247, 0.3421, 0.3095\}$ . One can verify that both  $\mathbf{x} = \{0.9247, 0.3421, 0\}$  and  $\mathbf{x} = \{0.9247, 0, 0.3095\}$  are both equilibria.

### $\mathcal{M}_4$ : Proportional Allocation, Continuous Action Space

We first prove the existence of PNE and then present an algorithm to search the PNE for the mechanism  $\mathcal{M}_4$ .

**Theorem 1** *For the full-information setting, there exists a PNE for the mechanism  $\mathcal{M}_4$ .*

**Proof.** Consider an action profile  $\{x_i\}_{i \in [N]}$ . Denote  $x_{-i} = \sum_{j \neq i} x_j$ . If  $\sum_{i=1}^N x_i > 0$ , the utility of user  $i$  is

$$u_i(x_i, x_{-i}) = R \frac{x_i}{x_i + x_{-i}} - \frac{x_i}{q_i} c, \quad (1)$$

constrained by  $0 \leq x_i \leq q_i$ .

The first order derivative of  $u_i$  w.r.t  $x_i$  is

$$\frac{\partial u_i}{\partial x_i} = R \frac{x_{-i}}{(x_i + x_{-i})^2} - \frac{c}{q_i}. \quad (2)$$

By setting the above derivative to zero, we get the best response strategy of user  $i$ :

$$x_i(q_i, x_{-i}) = \sqrt{\frac{Rq_i x_{-i}}{c}} - x_{-i}. \quad (3)$$

We have the following observations for this best response strategy:

1. Clearly  $x_i = 0$  is not the best response when  $x_{-i} = 0$  because user  $i$  would not be rewarded with  $x_i = 0$ . Actually there is not a best response for user  $i$  when  $x_{-i} = 0$ : if she gets a positive utility by contributing  $\delta > 0$ , she will profitably deviate by contributing  $\frac{\delta}{2}$ . Therefore zero-contribution ( $x_i = 0, \forall i$ ) is not an equilibrium strategy but it is a fixed point for Eqn. (3).
2.  $x_i$  calculated from Eqn. (3) can be smaller than zero or larger than  $q_i$ , which is not a feasible action. If  $x_i(q_i, x_{-i}) < 0$ , it means that Eqn. (2) will be smaller than zero when  $x_i > 0$ , so it is better to make zero contribution. If  $x_i(q_i, x_{-i}) \geq q_i$ , Eqn. (2) will be larger than 0, so  $u_i(x_i, x_{-i})$  increases w.r.t.  $x_i$ . Therefore it is better to contribute  $q_i$ .

Based on the two observations, we consider a perturbed game (Feldman, Lai, and Li 2009), in which the action of each user is lower bounded by a small positive quality  $\epsilon$  and the best response strategy of  $i$  is as below.

$$x_i^*(q_i, x_{-i}, \epsilon) = \begin{cases} \epsilon & \text{if } x_i(q_i, x_{-i}) \leq \epsilon \\ x_i(q_i, x_{-i}) & \text{if } \epsilon < x_i(q_i, x_{-i}) < q_i \\ q_i & \text{if } x_i(q_i, x_{-i}) \geq q_i \end{cases} \quad (4)$$

where  $x_i(\cdot, \cdot)$  is defined in Eqn. (3). In the remaining part of the proof, we show that (1) there exists a PNE for the perturbed game; (2) as  $\epsilon \rightarrow 0$ , any PNE of the perturbed game will not converge to zero contribution point and (3) by setting  $\epsilon \rightarrow 0$ , we get the PNE for the original game.

Denote the space  $[0, q_1] \times [0, q_2] \times \dots \times [0, q_N]$  as  $X$ , which is convex and compact. Define a mapping  $f$  from  $X$  to itself,

in which for any fixed  $\epsilon, \forall i \in [N], f_i(\mathbf{x}, \epsilon) = x_i^*(q_i, x_{-i}, \epsilon)$ . It is easy to verify that  $f$  is a continuous mapping. According to Brouwer fixed-point theorem (Border 1989), we know  $f$  has at least one fixed-point in  $X$ , which is the equilibrium of the perturbed game. Denote one fixed point as  $\mathbf{x}^\epsilon$ . Since  $X$  is compact, we could always find a series of  $\{\epsilon_n\} \rightarrow 0$  with their corresponding  $\mathbf{x}^{\epsilon_n}$  converging. Denote the limit point as  $\mathbf{x}^0$ .

Next we show  $\mathbf{x}^0$  is not the zero contribution point. Otherwise,  $\sum_{i=1}^N x_i^{\epsilon_n}$  could be infinitely close to zero as  $n \rightarrow \infty$ . What's more, both  $x_1^{\epsilon_n}$  and  $x_2^{\epsilon_n}$  are strictly less than the  $q_1$  and  $q_2$  respectively. We set  $\sum_{i=3}^N x_i^{\epsilon_n} = \delta_n$  and  $Q = \frac{c}{q_1} + \frac{c}{q_2}$ . By Eqn. (3) we obtain:

$$\begin{aligned} \sqrt{\frac{Rq_1(\delta_n + x_2^{\epsilon_n})}{c}} - (\delta_n + x_2^{\epsilon_n}) &= x_1^{\epsilon_n} \\ \sqrt{\frac{Rq_2(\delta_n + x_1^{\epsilon_n})}{c}} - (\delta_n + x_1^{\epsilon_n}) &= x_2^{\epsilon_n}. \end{aligned} \quad (5)$$

Add up the two equations in Eqn. (5) and solve it, the positive root is

$$x_1^{\epsilon_n} + x_2^{\epsilon_n} = \frac{1}{2} \left( \frac{R}{Q} + \sqrt{\left( \frac{R}{Q} \right)^2 + 4\delta_n \frac{R}{Q}} \right) - \delta_n, \quad (6)$$

which will not tend to zero as  $n \rightarrow \infty$ . This contradicts with the assumption that  $\sum_{i=1}^N x_i^{\epsilon_n}$  could be infinitely close to zero as  $n \rightarrow \infty$ .

Finally we prove  $\mathbf{x}^0$  is the equilibrium strategy of the original game by contradiction. Three possible alternative cases need discussing. They are: user  $j$  would like to deviate from

- $x_j^0 = 0$  to  $x_j' > 0$ ;
- $x_j^0 \in (0, q_j)$  to some  $x_j'$  in  $[0, q_j] - \{x_j^0\}$ ;
- $x_j^0 = q_j$  to  $x_j' < q_j$ .

We just discuss the first case here. Similar method could be applied to prove other cases and we put them in the appendix. Suppose user  $j$  could profitably deviate by contributing a content with quality no less than  $\delta (> 0)$ . Then we obtain

$$\sqrt{\frac{Rq_j x_{-j}^0}{c}} - x_{-j}^0 \geq \delta. \quad (7)$$

Since  $x_j^0 = 0$ , we know given any sufficiently small positive  $\epsilon, \exists N', \forall n > N', x_j^{\epsilon_n} < \epsilon$ , i.e.,

$$\sqrt{\frac{Rq_j x_{-j}^{\epsilon_n}}{c}} - x_{-j}^{\epsilon_n} < \epsilon. \quad (8)$$

$\mathbf{x}^{\epsilon_n} \rightarrow \mathbf{x}^0$  implies  $x_{-j}^{\epsilon_n} \rightarrow x_{-j}^0$ . But as  $\epsilon \rightarrow 0$ , we can verify  $x_{-j}^{\epsilon_n}$  will not converge to  $x_{-j}^0$  by Eqn. (7) and Eqn. (8). This contradicts with  $\mathbf{x}^{\epsilon_n} \rightarrow \mathbf{x}^0$ . Details could be found in the appendix.

Therefore, there exists a PNE for the mechanism  $\mathcal{M}_4$ . ■

Given the existence of PNE, we can obtain the following lemmas about the properties of an equilibrium profile, which will be used to find a PNE strategy. The proofs of the first two lemmas can be found in Appendix.

<sup>2</sup>We denote vectors with bold face letters in this paper.

**Lemma 2** Consider two users with  $q_i \leq q_j$ . If  $x_i = q_i$  holds in an equilibrium, then  $x_j = q_j$  holds in the same equilibrium.

Given Mechanism  $\mathcal{M}_4$ , in which  $N$  users compete for the reward  $R$ , we can induce a *local game* with  $m$  users: only the first  $m$  users compete for the reward  $R$ . As shown in the following lemma, an equilibrium of the induced local game can be connected to the equilibrium of the original game (i.e., Mechanism  $\mathcal{M}_4$  with  $N$  users) under certain condition.

**Lemma 3** If  $\{x_i\}_{i \in [m]}$  is an equilibrium of the induced local game with the first  $m$  users and  $\sum_{i=1}^m x_i \geq \frac{Rq_{m+1}}{c}$ , then  $\{y_i\}_{i \in [N]}$ , where  $y_i = x_i, \forall 1 \leq i \leq m$  and  $y_i = 0, \forall m+1 \leq i \leq N$ , is an equilibrium of the original game.

**Proof.** If user  $j (\geq m+1)$  deviates by contributing  $z > 0$ , her utility is

$$\begin{aligned} u_j &= \frac{Rz}{\sum_{i=1}^m x_i + z} - \frac{z}{q_j} < z \left( \frac{R}{\sum_{i=1}^m x_i} - \frac{c}{q_j} \right) \\ &\leq z \left( \frac{R}{\sum_{i=1}^m x_i} - \frac{c}{q_{m+1}} \right) \\ &= \frac{cz}{q_{m+1} \sum_{i=1}^m x_i} \left( \frac{Rq_{m+1}}{c} - \sum_{i=1}^m x_i \right) \leq 0. \end{aligned} \quad (9)$$

That is, user  $j (j \geq m+1)$  cannot be better off by unilaterally deviating. The original game is the same as the local game for the first  $m$  users, and they will not deviate unilaterally. Therefore  $\{y_i\}_{i \in [N]}$  is an equilibrium of the original game. ■

Based on the previous lemmas, we propose Algorithm 1 to find a PNE for a local game induced from the original game and verify whether it is a PNE of the original game by Lemma. 3. Then we discuss how to find a PNE for mechanism  $\mathcal{M}_4$  using Algorithm 1.

**Algorithm 1** The algorithm to find a PNE of the original game from an induced local game

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**Input:**  $\mathbf{q} = (q_1, q_2, \dots, q_n)$  where  $q_1 \geq q_2 \geq \dots \geq q_n$ ;

**Output:**  $\mathbf{x}^{(n)} = (x_1^{(n)}, x_2^{(n)}, \dots, x_n^{(n)})$

- 1: Calculate the  $y_{ni} \forall i \in [n]$  with Eqn. (14). If none of the  $y_{ni}$  is smaller than zero or larger than the corresponding  $q_i$ ,  $\mathbf{x}^{(n)} \leftarrow \mathbf{y}_n$ ; verify whether it is a PNE of the original game by Lemma. 3; if so, return  $\mathbf{x}^{(n)}$ .
- 2: **for**  $m \leftarrow 1 : n$  **do**
- 3: Calculate  $x_{nim} \forall i \in \{m+1, \dots, n\}$  with Eqn. (18).  $x_i^{(n)} \leftarrow q_i \forall i \in \{1, \dots, m\}, x_i^{(n)} \leftarrow x_{nim} \forall i \in \{m+1, \dots, n\}$  if they are all feasible;
- 4: **if**  $\mathbf{x}^{(n)}$  is a local PNE (verified by Lemma.4) **then**
- 5: Verify whether  $\mathbf{x}^{(n)}$  is a PNE of the original game and return it if so;
- 6: **end if**
- 7: **end for**

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**Lemma 4** In Algorithm 1,  $\mathbf{x}^{(n)}$  is a PNE of the local induced game with  $n$  users if the following condition holds:



- If  $m < n$ ,

$$\frac{Rq_1}{2c}(1 - \sqrt{1 - \frac{4c}{R}}) \leq \sum_{k=1}^n x_i^{(n)} \leq \frac{Rq_m}{2c}(1 + \sqrt{1 - \frac{4c}{R}});$$

- If  $m = n$ ,

$$R \geq \frac{cQ_m^2}{q_i(Q_m - q_i)} \forall i \in \{1, m\}.$$

**Proof.** If user  $i$  contributes her type in equilibrium, we know the solution of

$$\sqrt{\frac{Rq_i x_{-i}}{c}} - x_{-i} \geq q_i$$

is non-empty. And we can easily infer that  $R \geq 4c$  is a necessary condition for that. We temporarily denote  $\sum_{k=1}^n x_i^{(n)}$  as  $x$ .

When  $m < n$ ,  $x_j^{(n)} < q_j$  holds  $\forall j \in \{m+1, \dots, n\}$ . They could be regarded as the best response strategies without constraints. If anyone of the top  $m$  users does not want to change her strategy unilaterally, the following inequality holds  $\forall i \in [m]$ :

$$\sqrt{\frac{Rq_i(x - q_i)}{c}} - (x - q_i) \geq q_i \quad (10)$$

Eqn. (10) suggests that

$$\frac{Rq_i}{2c}(1 - \sqrt{1 - \frac{4c}{R}}) \leq x \leq \frac{Rq_i}{2c}(1 + \sqrt{1 - \frac{4c}{R}}) \forall i \in [m],$$

i.e.

$$\frac{Rq_1}{2c}(1 - \sqrt{1 - \frac{4c}{R}}) \leq x \leq \frac{Rq_m}{2c}(1 + \sqrt{1 - \frac{4c}{R}}). \quad (11)$$

When  $m = n$ , we know all of the  $n$  people contribute their types. If nobody could profitably deviate unilaterally, we obtain  $\forall i \in [m]$ ,

$$\sqrt{\frac{Rq_i(Q_m - q_i)}{c}} - (Q_m - q_i) \geq q_i, \quad (12)$$

i.e.

$$R \geq \frac{cQ_m^2}{q_i(Q_m - q_i)} \forall i \in \{1, m\} \quad (13)$$

■

The functions used in Algorithm 1 are listed from Eqn. (14) to Eqn. (18) and their derivations are in the appendix.

$$y_{ni} = \frac{R(n-1)}{c \sum_{k=1}^n \frac{1}{q_k}} [1 - \frac{n-1}{q_i \sum_{k=1}^n \frac{1}{q_k}}] \quad (14)$$

$$Q_m = \sum_{i=1}^m q_i \quad (15)$$

$$A_{nm} = \sum_{i=m+1}^n \frac{c}{Rq_i} \quad \forall n \leq N, n \geq m+1 \quad (16)$$

$$s_{nm} = \frac{(n-m-1) + \sqrt{(n-m-1)^2 + 4Q_m A_{nm}}}{2A_{nm}} \quad (17)$$

$$x_{nim} = s_{nm} - \frac{c}{Rq_i} s_{nm}^2 \quad (18)$$

**Theorem 5** Recursively calling Algorithm 1 from  $n = 2$  to  $N$ , it outputs a PNE of the original game.

**Proof.** By Theorem 1, the induced local game has a PNE.

Given an  $n \leq N$ , if the algorithm stops at Step 1, then  $x^{(n)}$  is a PNE of the original game.

If the algorithm does not stop at Step 1, given the top  $m$  people contributing their types, Eqn. (18) could be seen as the strategies that people  $m+1, \dots, n$  do not want to deviate if they are feasible. If the top  $m$  people do not want to deviate neither, we find a PNE of the induced local game. Lemma 2 describes the structure of all the PNEs, and therefore Algorithm 1 will traverse all the local PNEs of the induced game. Further, we note that a PNE of the original game is also a PNE of some induced local game. So the PNE of the original game could certainly be found by Algorithm 1. ■

## Partial-information Setting

In this section, we investigate the existence of pure Nash equilibrium of UGC mechanisms under the partial-information setting. We discuss three mechanisms here and leave another one (because of its difficulty) to the future work.

### $\mathcal{M}_5$ : Top $K$ Allocation, Binary Action Space

For simplicity, we only consider the case that  $R > Kc$  and omit the marginal case  $R \leq Kc$  here.

Let function  $T(x)$  denote the probability that a user with quality  $x$  is in one of the top  $K$  contributors. Clearly, if  $N - K \leq 0$ , we have  $T(x) = 1$ ; when  $N - K \geq 1$ , we have

$$T(x) = \sum_{j=0}^{K-1} \binom{N-1}{j} F(x)^{N-1-j} (1-F(x))^j. \quad (19)$$

Intuitively, a user with higher quality and fewer competitors is more likely to get the reward:

**Lemma 6**  $T(x)$  is a non-decreasing function of  $x$ .

**Proof.** We only need to discuss the non-trivial case, i.e.  $N - K \geq 1$ .

$$\begin{aligned} & \frac{\partial T(x)}{\partial x} \\ &= f(x) \left\{ (N-1)F(x)^{N-2} + \sum_{j=N-2}^{N-K} \binom{N-1}{j} \right. \\ & \quad \left[ jF(x)^{j-1}(1-F(x))^{N-1-j} \right. \\ & \quad \left. - (N-1-j)F(x)^j(1-F(x))^{N-2-j} \right] \Big\} \\ &= (N-1)f(x) \binom{N-2}{K-1} F(x)^{N-K-1} (1-F(x))^{K-1} \\ &\geq 0 \end{aligned} \quad (20)$$

Thus  $T(x)$  is a non-decreasing function of  $x$ . ■

Then we can construct a symmetric cut-off equilibrium (Fudenberg and Tirole 1991) for  $\mathcal{M}_5$ :

**Theorem 7** Denote the unique root of the following equation as  $x^*$ .

$$\frac{R}{K}T(x) - c = 0 \quad (21)$$

$\forall i \in [N]$ , we have that

$$\beta(q_i) = \begin{cases} q_i & \text{if } q_i \geq x^* \\ 0 & \text{if } q_i < x^* \end{cases} \quad (22)$$

is an equilibrium strategy of  $\mathcal{M}_5$ .

**Proof.** Following the strategy, if a user with type  $q(\geq x^*)$  chooses to participate in the game, the probability that she could get the reward is

$$\begin{aligned} P(q) &= \sum_{n=0}^{N-K-1} \binom{N-1}{n} F(x^*)^n \\ &\quad \sum_{j=0}^{K-1} \binom{N-n-1}{j} (F(q) - F(x^*))^{N-1-n-j} (1 - F(q))^j \\ &\quad + \sum_{n=N-K}^{N-1} \binom{N-1}{n} F(x^*)^n (1 - F(x^*))^{N-1-n}. \end{aligned} \quad (23)$$

If  $q > x^*$ , we have  $P(q) > \frac{cK}{R}$ , and the user's expected utility if she participates in the game is

$$\frac{R}{K}P(q) - c > \frac{R}{K} \frac{cK}{R} - c = 0. \quad (24)$$

If  $q \leq x^*$ , the expected utility for the user is 0. Thus none can be better off by deviating her strategy unilaterally. ■

### $\mathcal{M}_6$ : Top $K$ Allocation, Continuous Action Space

We first give a general description to a symmetric equilibrium strategy, then prove the existence of PNE when  $F$  is the uniform distribution with the method of (Krishna 2009), and finally generalize the result to any general distribution.

Let us consider a symmetric strategy  $\beta(\cdot)$ : each user  $i$  with quality  $q_i$  will contribute to the site with quality  $\beta(q_i)$ .

**Lemma 8** If  $\beta(\cdot)$  is an equilibrium strategy, then  $q_i > 0 \Rightarrow \beta(q_i) > 0$ .

**Proof.** First we declare that given  $\beta(\cdot)$  is an equilibrium strategy, if  $q_i < q_j$  and  $\beta(q_i) > 0$ , then we have  $\beta(q_j) > 0$ . Otherwise, user  $j$  can take the action  $x_j = \beta(q_i) + \delta$ , where  $\delta$  is sufficiently small, to get positive utility, which contradicts with that  $\beta(\cdot)$  is an equilibrium strategy.

Therefore, if there exists some  $q > 0$  such that  $\beta(q) = 0$ , then we have  $\beta(x) = 0 \forall x \in [0, q]$ . For any user whose type falls in  $[0, q]$ , if she contributes  $\epsilon$ , then her expected utility is

$$\frac{R}{K} \sum_{n=N-K}^{N-1} \binom{N-1}{n} F(q)^n (1 - F(q))^{N-n-1} - \frac{\epsilon}{q_i} c \quad (25)$$

We can always find an  $\epsilon$  small enough to ensure the equation above is larger than 0. Therefore  $\beta(q_i)$  is not an equilibrium, which leads to a contradiction. Thus, there does not exist a  $q > 0$  such that  $\beta(q) = 0$ . ■

Suppose that users  $j \neq i$  follow the symmetric equilibrium strategy  $\beta(\cdot)$ . If user  $i$  pretends that her type is  $x$  and contributes  $\beta(x)$ , her expected utility is

$$u_i(x; q_i) = \frac{R}{K}T(x) - \frac{\beta(x)}{q_i}c. \quad (26)$$

The first order derivative of  $u_i$  is

$$\frac{\partial u_i(x; q_i)}{\partial x} = \frac{R}{K} \frac{\partial T(x)}{\partial x} - \frac{c\beta'(x)}{q_i}. \quad (27)$$

If  $\beta(q_i)$  is an equilibrium strategy for user  $i$ , her expected utility should be maximized at  $x = q_i$ . That is, we should have

$$\left. \frac{\partial u_i(x; q_i)}{\partial x} \right|_{x=q_i} = 0.$$

Note that  $\beta(0) = 0$ . Solving the above equation, we get

$$\beta(x) = \frac{R(N-1)}{cK} \binom{N-2}{K-1} \int_0^x t F(t)^{N-K-1} (1 - F(t))^{K-1} dt F(t) \quad (28)$$

Then we have the following results.

**Lemma 9** If  $\beta(x) \leq x, \forall x \in [0, 1]$ , then the function  $\beta(\cdot)$  in the above equation is an equilibrium strategy.

However, it is possible that  $\beta(x)$  expressed by Eqn. (28) is larger than  $x$ . For example, if  $F$  is the uniform distribution over  $[0, 1]$ ,  $\beta(x)$  can be written as below.

$$\beta(x) = \frac{R}{cK} (N-1) \binom{N-2}{K-1} \sum_{k=0}^{K-1} (-1)^{K-k-1} \binom{K-1}{k} \frac{x^{N-k}}{N-k} \quad (29)$$

Then we have<sup>3</sup>

$$\begin{aligned} \beta(1) &= \int_0^1 \frac{Rx}{cK} \frac{\partial T(x)}{\partial x} dx \\ &= \int_0^1 \frac{R}{cK} (N-1) \binom{N-2}{K-1} \int_0^1 x^{N-K} (1-x)^{K-1} dx \\ &= \frac{R}{cK} (N-1) \binom{N-2}{K-1} B(N-K+1, K) \\ &= \frac{R}{cK} \frac{N-K}{N}, \end{aligned}$$

which might be larger than 1.

If  $\beta(x) > x$  for some  $x \in [0, 1]$ ,  $\beta(x)$  will not be an equilibrium strategy anymore. We need to calibrate  $\beta(x)$ . For ease of description, we first illustrate how to make calibration when  $F$  is the uniform distribution, and then extend to a general distribution.

With some derivations, one can get that the equation  $\beta(x) = x$  has at most positive two solutions in the region  $(0, 1]$  for uniform distribution  $F$ . If there exist two positive solutions (denote them as  $x_1$  and  $x_2$ , and assume  $x_1 < x_2$ ), there will be an  $x_p (> x_1)$  that satisfies  $\beta'(x_p) = 1$ . Then we have:

<sup>3</sup> $B(\cdot, \cdot)$  is the beta function.

**Theorem 10** If  $F$  is the uniform distribution over  $[0, 1]$  and  $\beta(x) = x$  has two solutions in  $(0, 1]$ , the following  $\beta^*(\cdot)$  function is an equilibrium, where  $x_1$  and  $x_p$  are defined above.

$$\beta^*(x) = \begin{cases} \beta(x) & x \in [0, x_1] \\ x & x \in (x_1, x_p] \\ \beta(x) - \beta(x_p) + x_p & x \in (x_p, 1] \end{cases} \quad (30)$$

**Proof.** First, if  $x \in [0, x_1]$ , since  $\beta(x) \leq x$ , we have that  $\beta^*(x) = \beta(x)$  is the best response of type  $x$ .

Second, it is clear that the first order derivative  $\beta'(x)$  is larger than 1 for any  $x \in (x_1, x_p)$ . Suppose that all the others follow strategy  $\beta^*(\cdot)$  except user  $i$ , and suppose she pretends that her type is  $x$ .

- If  $x \in (x_1, x_p)$ , we have

$$u_i(x; q_i) = \frac{R}{K}T(x) - \frac{x}{q_i}c,$$

and

$$\frac{\partial u_i(x; q_i)}{\partial x} > \frac{R}{K} \frac{\partial T(x)}{\partial x} - \frac{\beta'(x)}{q_i}c \geq \frac{R}{K} \frac{\partial T(x)}{\partial x} - \frac{\beta'(x)}{x}c = 0.$$

Therefore, the larger  $x$  is, the larger utility she will get. However, since the contributed quality is upper bounded by her type  $q_i$ , the best choice for her is to take the action  $x_i = q_i$ .

- If  $x \in [0, x_1]$ , we have

$$\frac{\partial u_i(x; q_i)}{\partial x} = \frac{R}{K} \frac{\partial T(x)}{\partial x} - \frac{\beta'(x)}{q_i}c > \frac{R}{K} \frac{\partial T(x)}{\partial x} - \frac{\beta'(x)}{x}c = 0.$$

So she should pretend her type is  $x_1$ , which is still worse than revealing the true type  $q_i$ .

Thus, for any  $x$  in  $(x_1, x_p]$ , the best response is  $\beta^*(x) = x$ .

Third, note that for any  $x$  in  $(x_p, 1]$ , we have  $\beta'(x) \leq 1$ . Integrating  $\beta'(x)$  from  $x_p$  to  $x$  and using  $\beta(x_p) = x_p$ , we get

$$\beta^*(x) - x_p = \beta(x) - \beta(x_p).$$

It is easy to verify that  $\beta^*(x) \leq x$  for any  $x$  in  $(x_p, 1]$ . Therefore, we get that  $\beta^*(x) = \beta(x) + x_p - \beta(x_p)$  is the best response for any  $x$  in  $(x_p, 1]$ .

Thus, the theorem is proved. ■

Figure 1 shows an equilibrium strategy for  $N = 11$ ,  $K = 5$ ,  $c = 1$ ,  $R = 8$ .

Next we generalize the above results. For a general distribution  $F$  over  $[0, 1]$ , we first initialize  $\beta^*(x) = \beta(x)$ ,  $\forall x \in [0, 1]$  and then calibrate  $\beta^*(x)$  as follows.

1. Check whether  $\beta^*(x) > x$  starting from  $x = 0$  to  $x = 1$ .
2. Suppose  $[x_1, x_2]$  is the first interval that  $\beta^*(x) > x$ , and  $x_p$  is the point in this interval satisfying  $\beta'(x_p) = 1$ . Let  $o$  denote the value of  $\beta^*(x)$  at  $x_p$  (i.e.,  $o = \beta^*(x_p)$ ), and then calibrate  $\beta^*(x) = x$ ,  $\forall x \in [x_1, x_p]$  and  $\beta^*(x) = \beta^*(x) - o + x_p$ ,  $\forall x \in (x_p, 1]$ .
3. Continue to check whether  $\beta^*(x) > x$  starting from  $x = x_p$  to  $x = 1$ . If there is still some interval with  $\beta^*(x) > x$ , we calibrate  $\beta^*(x)$  as shown in Step 2.

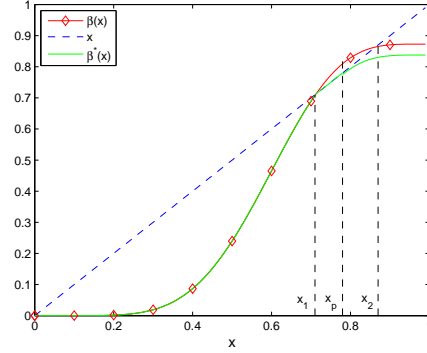


Figure 1: An example equilibrium strategy

4. We repeat the checking and calibrating procedure until  $\beta^*(x) \leq x$ ,  $\forall x \in [0, 1]$ .

After this calibration process, we obtain an equilibrium strategy  $\beta^*(x)$  from  $\beta(x)$ , which is shown in Eqn. (28), for any distribution  $F$ . Therefore we have the following theorem.

**Theorem 11**  $\mathcal{M}_6$  has at least one symmetric PNE.

#### $\mathcal{M}_7$ : Proportional Allocation, Binary Action Space

Now we study the existence of PNE of the mechanism with the proportional allocation rule and the binary action space under the partial-information setting.

For user  $i$ , let us consider the following cut-off strategy:

$$\beta_i(x) = \begin{cases} q_i & \text{if } x \geq x^* \\ 0 & \text{if } x < x^*, \end{cases} \quad (31)$$

where  $x^*$  is a threshold parameter.

Suppose that users  $j \neq i$  follow the above strategy. Then the expected utility of user  $i$  can be written as follows if she participate in the game ( $x_i = q_i$ ).

$$u_i(q_i; x^*) = \sum_{k=0}^{N-1} \binom{N-1}{k} F(x^*)^{N-1-k} (1-F(x^*))^k u_i(q_i, k; x^*) - c \stackrel{\text{def}}{=} y(q_i, x^*) - c, \quad (32)$$

where  $u_i(q_i, k; x^*)$  is the expected utility of user  $i$  given another  $k$  users with quality larger than  $x^*$  participating in the game, and it can be written as

$$R \int_{x^*}^1 \dots \int_{x^*}^1 \frac{q_i}{q_i + t_1 + t_2 + \dots + t_k} dF(t_1|x^*) \dots dF(t_k|x^*).$$

In Eqn. (32), when  $k = 0$ ,  $u_i(q_i, k; x^*) = R$ , which means that user  $i$  gets all the reward  $R$  since no other user participate in the game ( $k = 0$  means  $x_j = 0$ ,  $\forall j \neq i$ ).

If Eqn. (31) is an equilibrium strategy, then the best response of user  $i$  is also to follow the strategy given that all other users follow the strategy. That is,

$$u_i(q_i; x^*) = \begin{cases} > 0 & \text{if } q_i \geq x^* \\ = 0 & \text{if } q_i = x^* \\ < 0 & \text{if } x < x^* \end{cases} \quad (33)$$

It is not difficult to get that

- $y(q_i, x^*)$  increases w.r.t.  $q_i$ ,
- $y(0, 0) - c = -c < 0$  and  $y(1, 1) - c = R - c > 0$ .

We further assume that  $F(\cdot)$  is a continuous function; consequently,  $y(t, t)$  is continuous. Therefore, there exists an  $x^*$  satisfying the three conditions in Eqn. (33); in turn, this  $x^*$  makes Eqn. (31) a (symmetric) equilibrium strategy. Thus we have the following theorem.

**Theorem 12**  $\mathcal{M}_7$  has at least one PNE if  $R > c$ .

## Conclusions and Future work

We studied UGC mechanisms under a new framework: Users are heterogeneous and the best quality a user can contribute can be different from others. Under the framework, we considered several mechanisms involving two allocation rules, two action spaces and two information settings. We proved the existence of multiple PNE for some mechanisms, the existence and uniqueness of PNE for some mechanisms, and the non-existence of PNE for some other mechanisms.

There are many issues to explore about UGC mechanisms in the future. First, the efficiency analysis is a meaningful topic given the existence of multiple equilibria for some mechanisms. Second, we plan to study the mixed Nash equilibrium for UGC mechanisms. Third, we only considered linear cost function in this work. We will investigate more general cost functions (e.g., concave functions). Fourth, the comparison between different mechanisms would be an interesting topic.

## References

- [Anderson et al. 2013] Anderson, A.; Huttenlocher, D.; Kleinberg, J.; and Leskovec, J. 2013. Steering user behavior with badges. In *Proceedings of the 22nd international conference on World Wide Web*, WWW '13, 95–106.
- [Border 1989] Border, K. 1989. *Fixed point theorems with applications to economics and game theory*. New York, NY, USA: Cambridge University Press.
- [Cachon and Lariviere 1999] Cachon, G. P., and Lariviere, M. A. 1999. An equilibrium analysis of linear, proportional and uniform allocation of scarce capacity. *IIE Transactions* 31(9):835–849.
- [Chawla, Hartline, and Sivan 2012] Chawla, S.; Hartline, J. D.; and Sivan, B. 2012. Optimal crowdsourcing contests. In *Proceedings of the Twenty-Third Annual ACM-SIAM Symposium on Discrete Algorithms*, SODA '12, 856–868.
- [Chen 2009] Chen, D. Y.-C. 2009. *Essays on mobile advertising and commerce*. Ph.D. Dissertation, Harvard University Cambridge, Massachusetts.
- [Easley and Ghosh 2013] Easley, D., and Ghosh, A. 2013. Incentives, gamification, and game theory: an economic approach to badge design. In *Proceedings of the fourteenth ACM conference on Electronic commerce*, EC '13, 359–376.
- [Feldman, Lai, and Li 2009] Feldman, M.; Lai, K.; and Li, Z. 2009. The proportional-share allocation market for computational resources. *IEEE Trans. Parallel Distrib. Syst.* 20(8):1075–1088.

- [Fudenberg and Tirole 1991] Fudenberg, D., and Tirole, J. 1991. *Game theory*. MIT press.
- [Ghosh and Hummel 2011] Ghosh, A., and Hummel, P. 2011. A game-theoretic analysis of rank-order mechanisms for user-generated content. In *Proceedings of the 12th ACM conference on Electronic commerce*, EC '11, 189–198.
- [Ghosh and Kleinberg 2013] Ghosh, A., and Kleinberg, J. 2013. Incentivizing participation in online forums for education. In *Proceedings of the fourteenth ACM conference on Electronic commerce*, EC '13, 525–542.
- [Ghosh and McAfee 2011] Ghosh, A., and McAfee, P. 2011. Incentivizing high-quality user-generated content. In *Proceedings of the 20th international conference on World wide web*, WWW '11, 137–146.
- [Ghosh 2012] Ghosh, A. 2012. Social computing and user-generated content: a game-theoretic approach. *SIGecom Exch.* 11(2):16–21.
- [Jain, Chen, and Parkes 2009] Jain, S.; Chen, Y.; and Parkes, D. C. 2009. Designing incentives for online question and answer forums. In *Proceedings of the 10th ACM conference on Electronic commerce*, EC '09, 129–138.
- [Kelly, Maulloo, and Tan 1998] Kelly, F.; Maulloo, A.; and Tan, D. 1998. Rate control in communication networks: shadow prices, proportional fairness and stability. In *Journal of the Operational Research Society*.
- [Kelly 1997] Kelly, F. 1997. Charging and rate control for elastic traffic. *European transactions on Telecommunications* 8(1):33–37.
- [Krishna 2009] Krishna, V. 2009. *Auction theory*. Academic press.
- [Nisan et al. 2007] Nisan, N.; Roughgarden, T.; Tardos, E.; and Vazirani, V. V. 2007. *Algorithmic Game Theory*. New York, NY, USA: Cambridge University Press.
- [Sandholm and Lai 2010] Sandholm, T., and Lai, K. 2010. Dynamic proportional share scheduling in hadoop. In *Job scheduling strategies for parallel processing*, 110–131. Springer.
- [Stoica et al. 1996] Stoica, I.; Abdel-Wahab, H.; Jeffay, K.; Baruah, S.; Gehrke, J.; and Plaxton, C. 1996. A proportional share resource allocation algorithm for real-time, time-shared systems. In *Real-Time Systems Symposium, 1996., 17th IEEE*, 288–299.

## Appendix

### Omitted Proofs

In this section, we give some technical details that are omitted in the main paper.

### Proof of Lemma 2

**Proof.** The proof is by contradiction. Suppose user  $i$  has a larger type than user  $j$  ( $q_i \geq q_j$ ) and contributes  $x_i < q_i$  in equilibrium.

Setting  $A = \sum_{k \neq i, j} x_k$ , we have

$$\sqrt{\frac{Rq_i(A + q_j)}{c}} - (A + q_j) = x_i \quad (34)$$



and

$$\sqrt{\frac{Rq_j(A+x_i)}{c}} - (A+x_i) \geq q_j. \quad (35)$$

From Eqn. (34) and Eqn. (35) we get

$$q_j(A+x_i) \geq q_i(A+q_j). \quad (36)$$

With some simple derivations we can see

$$(q_j - q_i)A \geq q_j(q_i - x_i). \quad (37)$$

The left hand side of Eqn. (37) is smaller than zero but the right hand side is larger than zero. It is a contradiction. ■

#### Derivation of Eqn. (14)

By summing Eqn. (2) over  $i$ , we get

$$x_i + x_{-i} = \frac{R(N-1)}{c \sum_{k=1}^N \frac{1}{q_k}}. \quad (38)$$

Substituting the above equation to Eqn. (3), we obtain

$$x_i = \frac{R(N-1)}{c \sum_{k=1}^N \frac{1}{q_k}} \left[ 1 - \frac{N-1}{q_i \sum_{k=1}^N \frac{1}{q_k}} \right], \quad (39)$$

which is the best response without considering feasibility constraints.

#### Derivation of Eqn. (18)

Consider the case that only the first  $n$  users participating in the game and the first  $m$  users out of the  $n$  users contribute their types. For user  $i \in \{m+1, m+2 \dots n\}$ , Eqn. (2) can be written as

$$\frac{R(Q_m + y_{-i})}{(Q_m + y_i + y_{-i})^2} = \frac{c}{q_i} \quad \forall i \in \{m+1, m+2 \dots n\}. \quad (40)$$

Summing the above equation over  $i$ , we obtain

$$R \frac{(n-m)(Q_m + y_i + y_{-i}) - \sum_{i=m+1}^n y_i}{(Q_m + y_i + y_{-i})^2} = \sum_{i=m+1}^n \frac{c}{q_i}. \quad (41)$$

Solving the above equation, we get

$$y_i + y_{-i} = \frac{(n-m-1) - 2Q_m A_{nm} \pm \sqrt{(n-m-1)^2 + 4Q_m A_{nm}}}{2A_{nm}}.$$

We are only interested in the positive solution. Denoting

$$\begin{aligned} s_{nm} &= Q_m + \sum_{i=m+1}^n y_i \\ &= \frac{(n-m-1) + \sqrt{(n-m-1)^2 + 4Q_m A_{nm}}}{2A_{nm}}, \end{aligned} \quad (42)$$

we arrive at

$$y_i = s_{nm} - \frac{c}{Rq_i} s_{nm}^2. \quad (43)$$

#### Omitted proof of Theorem 1

Define

$$l(\delta, q) = \frac{1}{2} \left( \frac{Rq}{c} - 2\delta - \sqrt{\left( \frac{Rq}{c} \right)^2 - 4\delta \frac{Rq}{c}} \right) \quad (44)$$

and

$$u(\delta, q) = \frac{1}{2} \left( \frac{Rq}{c} - 2\delta + \sqrt{\left( \frac{Rq}{c} \right)^2 - 4\delta \frac{Rq}{c}} \right). \quad (45)$$

We find that

$$\begin{aligned} l(\delta, q) &= \frac{1}{2} \left( \frac{Rq}{c} - 2\delta - \sqrt{\left( \frac{Rq}{c} \right)^2 - 4\delta \frac{Rq}{c}} \right) \\ &= \frac{2\delta^2}{\frac{Rq}{c} - 2\delta + \sqrt{\left( \frac{Rq}{c} \right)^2 - 4\delta \frac{Rq}{c}}}. \end{aligned} \quad (46)$$

So  $l(\delta, q)$  monotonously increases with  $\delta$  and  $u(\delta, q)$  monotonously decreases with  $\delta$ .

If Eqn. (7) has no solution, we know that she will not change her strategy to generate a content with quality larger than  $\delta$ . We only consider the case that Eqn. (7) is solvable.

$$\begin{aligned} \sqrt{\frac{Rq_j x_{-j}^0}{c}} - x_{-j}^0 &\geq \delta \Rightarrow l(\delta, q_j) \leq x_{-j}^0 \leq u(\delta, q_j) \\ \sqrt{\frac{Rq_j x_{-j}^{\epsilon_n}}{c}} - x_{-j}^{\epsilon_n} &< \epsilon \Rightarrow x_{-j}^{\epsilon_n} < l(\epsilon, q_j) \text{ or } x_{-j}^{\epsilon_n} > u(\epsilon, q). \end{aligned} \quad (47)$$

As  $\epsilon \rightarrow 0$ ,  $\exists \delta_1 > 0, \delta_2 > 0$ , s.t.

$$\begin{aligned} l(\delta, q_j) - l(\epsilon, q_j) &\geq \delta_1 \\ u(\epsilon, q_j) - u(\delta, q_j) &\geq \delta_2, \end{aligned} \quad (48)$$

which shows that  $x_{-j}^{\epsilon_n}$  does not converge to  $x_{-j}^0$ . It is contradicted with  $x^{\epsilon_n} \rightarrow x^0$ .

If  $0 < x_j^0 < q_j$ , we know  $x_{-j}^{\epsilon_n}$  will converge to  $l(x_j^0, q_j)$  or  $u(x_j^0, q_j)$ . If user  $j$  wants to deviate to

- some another  $x'_j \in (0, q_j)$  but  $x_j^0 \neq x'_j$ , we know  $x_{-j}^0 = l(x'_j, q_j)$  or  $x_{-j}^0 = u(x'_j, q_j)$ , which contradicts with the fact that the series  $\{x^{\epsilon_n}\}$  converges.
- $x'_j = q_j$ , we obtain  $l(q_j, q_j) \leq x_{-j}^0 \leq u(q_j, q_j)$ . By the monotonicity of  $l(\cdot)$  and  $u(\cdot)$  we can find a contradiction.
- $x'_j = 0$ , we obtain  $x_{-j}^0 \leq l(0, q_j)$  or  $x_{-j}^0 \geq u(0, q_j)$ . Neither  $l(x_j^0, q_j)$  nor  $u(x_j^0, q_j)$  could fall in those regions.

If  $x_j^0 = q_j$  but she wants to deviate to some  $\delta' \leq \delta < q_j$ , we know  $x_{-j}^0 \leq l(\delta, q_j)$  or  $x_{-j}^0 \geq u(\delta, q_j)$ . While  $x_j^0 = q_j$  suggests that  $x_{-j}^{\epsilon_n}$  will converge to a point in the region  $l(q_i, q_i) \leq x_{-j}^0 \leq u(q_i, q_i)$ , so we find a contradiction.